MATH4210: Financial Mathematics Tutorial 10

Yi Shen

The Chinese University of Hong Kong

yishen@math.cuhk.edu.hk

20 November, 2024

Options

Q1 suppose $\exists\, t \in [0,T]$ satisfy $P_E(t,K) \leq Ke^{-r(T-t)} - S(t) \Rightarrow$

long 1 put option & borrow money $Ke^{-r(T-t)}$

long a stock

$\pi_1(t) = P_E(t,K) + S(t) - Ke^{-r(T-t)}$ ≤ 0

Proposition (Law of One Price)

If two portfolios have the same profit at maturity time T, then for all prior times $t < T$, the price of the portfolio's must be equal.

Proof.

By no-arbitrage, it is easy to prove by contradictions. □

Question 1

Show that the European put options with strike price K and maturity at time T satisfies $P_E(t,K) > Ke^{-r(T-t)} - S(t)$ for all $t < T$, where $S(t)$ is the stock price, r is the continuous compounded interest rate.

$\pi_1(T) = P_E(T,K) + S(T) - K$

$= (K - S_T)_+ + S(T) - K = \begin{cases} S_T - K & \text{if } S_T \geq K \\ 0 & \text{if } S_T < K \end{cases}$ $\Rightarrow$ positive

$\Rightarrow P(\pi_1(T) > 0) > 0.$

$\Rightarrow P(\pi_1(T) \geq 0) = 1$

$\Rightarrow$

Options

Assume $\exists t \le T_1$ $P_E(t,T_1) \geq P_E(t,T_2)$

π: short 1 $P_E(t,T_1)$

long 1 $P_E(t,T_2)$

$\pi(t) = P_E(t,T_2) - P_E(t,T_1) \leq 0$

$\pi(T_1) = P_E(T_1,T_2) - (K - S_{T_1}) \cdots ①$

$+ P_E(t,K) \geq Ke^{-r(T-t)} - S(t)$

$\cdots ②$

Question

Two vanilla put options are identical except for the maturity dates $T_1 < T_2$. If the interest rate is zero between T_1 and T_2, then $P_E(t,T_1) < P_E(t,T_2)$ at any time $t \leq T_1$.

$① + ② \Rightarrow \pi(T_1) \geq Ke^{-r(T_2-T_1)} - S_{T_1} - (K - S_{T_1})_+$

$= 0$

$\pi(T_1) \geq 0$

$\Rightarrow P(\pi(T_1) > 0) > 0$ $\Rightarrow$ exist arbitrage opp $\cdots$

$P(\pi(T_1) \geq 0) = 1$

Options

assume $\exists t \le T$

$$\begin{cases} 0 \geq P_E(t,K_2) - P_E(t,K_1) \\ P_E(t,K_2) - P_E(t,K_1) \geq (K_2-K_1)e^{-r(T-t)} \end{cases}$$

Π: long $P_E(t,K_1)$

short $P_E(t,K_2)$

long $(K_2-K_1)e^{-r(T-t)}$

Question

Suppose two put European options are identical except for the strike prices $0 < K_1 < K_2$, *show that*

$$0 < P_E(t,K_2) - P_E(t,K_1) < (K_2-K_1)e^{-r(T-t)},$$

at any time t before maturity T.

$\Pi(t) = (K_2-K_1)e^{-r(T-t)} - P_E(t,K_2) + P_E(t,K_1) \le 0$

$\Pi(T) = K_2 - K_1 - (K_2-S_T)_+ + (K_1-S_T)_+$

$$\begin{cases} S_T - S_T = 0 & \text{if } S_T < K_1 \\ S_T - K_1 > 0 & \text{if } K_1 \le S_T < K_2 \\ K_2 - K_1 > 0 & \text{if } S_T \geq K_2 \end{cases}$$

$P(\Pi(T) > 0) = 1 - P(S_T < K_1) > 0$

$P(\Pi(T) \geq 0) = 1$

Π_1 : long a $C_E(t,K)$

short $P_E(t,K)$

Π_2 : long $S(t)$

short $(Ke^{-r(T-t)} + De^{-r(t_D-t)})$

$\Pi_2(t_D) = S(t_D) - Ke^{-r(T-t_D)} - D + D$

$= S(t_D) - Ke^{-r(T-t_D)}$

$\Pi_2(T) = S(T) - K$

$\Pi_1(T) = (S_T - K)_+ - (K - S_T)_+ = \begin{cases} S_T - K & \text{if } S_T \geq K \\ S_T - K & \text{if } S_T < K \end{cases}$

Question (Put-Call Parity Relation with Dividend)

Prove the following. Assume that the value of the dividends of the stock paid during $[t, T]$ is a deterministic constant D at time $t_D \in (t, T]$. Let $S(t)$ be the stock price, r be the continuous compounding interest rate , $C_E(t, K)$ and $P_E(t, K)$ be the prices of European call and put option at time t with strike K and maturity T respectively. We have

$$C_E(t, K) - P_E(t, K) = S(t) - Ke^{-r(T-t)} - De^{-r(t_D-t)}$$

$\Pi_1(T) = \Pi_2(T) \quad \Rightarrow \Pi_1(t) = \Pi_2(t)$

Forward

Π_1 : long a $F(t,T)$, put $F(t,T)e^{-r(T-t)}$ in the bank.

Π_2 : long a stock $S(t)$

$\Pi_1(t) = F(t,T)e^{-r(T-t)}$

$\Pi_1(T) = F(t,T) + (\overset{\text{get } 1\, S_T}{S_T} - \overset{\text{pay}}{F(t,T)}) = S_T = \Pi_2(T)$

Question

Under no arbitrage opportunity assumptions and assume the continuous compounded interest rate is r, if the stock pays no dividend, show that $F(t,T) = S(t)e^{r(T-t)}$ *for* $t \cancel{\geq} \leq T$.

$\Rightarrow \Pi_1(t) = \Pi_2(t) = S_t = F(t,T)e^{-r(T-t)}$

$\Rightarrow F(t,T) = S_t e^{r(T-t)}$

Forward

π_1 : long a stock. S_0

$\pi_1(t) = S_t + dS_t = (1+d)S_t$

$\pi_1(T) = (1+d)S_T$

π_2 : long $(1+d)F(0,T)$, put $(1+d)F(0,T)e^{-rT}$ in the bank.

$\pi_2(0) = (1+d)F(0,T)e^{-rT}$

$\pi_2(T) = (1+d)F(0,T) + ((1+d)S_T - (1+d)F(0,T))$

$= (1+d)S_T = \pi_1(T)$

Question

Suppose the stock pay a dividend $d \times S(t)$ at time t, where $0 < t < T$ and $0 < d < 1$, show its forward price $F(0,T) = \frac{1}{1+d}S(0)e^{rT}$.

$S(0) = (1+d)F(0,T)e^{-rT}$

$\Rightarrow F(0,T) = \frac{1}{1+d}S(0)e^{rT}$